Chapter 9.1 – Simplify Radical Expressions

Any term under a radical sign is called a **radical** or a **square root expression**. The number or expression under the radical sign is called the radicand. The radicand is in simplest form if it contains no perfect square factors other than one.

To express a radical expression in simplest form we can use The Product Property of Square Roots.

**The Product Property of Square Roots** states: For any real numbers $a$ and $b$, where $a \geq 0$ and $b \geq 0$, \[\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} .\]

To simplify or to find another expression with the same value:

- Find two factors of the radicand either by finding the greatest perfect square factor or by finding a perfect square and then continuing to simplify.
- Use the Product Property of Square Roots
- Simplify the perfect square factor.

When simplifying radical expressions that contain variables, remember that positive numbers have two square roots, one positive and one negative.

If the exponent of the variable inside the radical is even and the simplified result is odd, use the absolute value to assure non-negative results.

\[
\begin{align*}
\sqrt{n^2} &= |n| \\
\sqrt{n^4} &= n^2 \\
\sqrt{n^6} &= |n^3| \\
\sqrt{n^8} &= n^4
\end{align*}
\]

This is because $a^m \cdot a^n = a^{m+n} = a^{2m}$

Therefore, $\sqrt{a^{2m}} = a^m$.

A variable to an even power is a perfect square.

**Examples:**

Express each in simplest radical form.

1. $\sqrt{400} = \sqrt{100} \cdot 4 = \sqrt{100} \cdot \sqrt{4} = 10 \cdot 2 = 20$
2. $-\sqrt{75} = -1 \cdot \sqrt{25} \cdot 3 = -1 \cdot \sqrt{25} \cdot \sqrt{3} = -1 \cdot 5 \cdot \sqrt{3} = -5\sqrt{3}$
3. $4\sqrt{a^2b^5c^6} = 4\sqrt{a^2} \sqrt{b^5} \sqrt{c^6}$
   \[4\sqrt{a^2} = 4|a|\]
   \[\sqrt{b^5} = \sqrt{b^4} \cdot b = b^2 \sqrt{b}\]
   \[\sqrt{c^6} = |c^3|\]
   \[= 4b^2|ac^3|\sqrt{b}\]
Chapter 9-2 – Add and Subtract Radical Expressions

To add or subtract Radical Expressions:

Step 1: **Simplify the radicals.**

Step 2: Combine like radicals.
- You can only add or subtract radicals together if they are **like radicals**.
- You add or subtract them in the same fashion that you do like terms. Combine the numbers that are in front of the like radicals and write that number in front of the like radical part.

Example 1: Add or subtract to simplify radical expression: \(2\sqrt{12} + \sqrt{27}\)

Solution:

Step 1: Simplify radicals

\[\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}\]
\[\sqrt{27} = \sqrt{9 \cdot 3} = \sqrt{9} \cdot \sqrt{3} = 3\sqrt{3}\]

Step 2: Combine like radicals

\[2\sqrt{12} + \sqrt{27} = 2 \cdot 2\sqrt{3} + 3\sqrt{3} = 4\sqrt{3} + 3\sqrt{3} = 7\sqrt{3}\]

Example 2: Add or subtract to simplify radical expression: \(3\sqrt{50} - 2\sqrt{8} - 5\sqrt{32}\)

Solution:

Step 1: Simplify radicals

\[\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}\]
\[\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}\]
\[\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}\]

Step 2: Combine like radicals

\[3\sqrt{50} - 2\sqrt{8} - 5\sqrt{32} = 3 \cdot 5\sqrt{2} - 2 \cdot 2\sqrt{2} - 5 \cdot 4\sqrt{2}\]

\[= 15\sqrt{2} - 4\sqrt{2} - 20\sqrt{2} = (15 - 4 - 20)\sqrt{2} = -9\sqrt{2}\]
Chapter 9-3 Multiply and Divide Radical Expressions

To multiply radical expressions you use the commutative and associative properties and the Product Property of Square Roots.

To Multiply Radical Expressions: Adding radical expressions: For any real number a and b where \( a \geq 0 \) and \( b \geq 0 \),
\[ a\sqrt{x} \cdot b\sqrt{y} = (a \cdot b)\sqrt{x \cdot y} \]

Example:
\[ \sqrt{15x} \cdot \sqrt{3x} = \]
\[ \sqrt{15x(3x^3)(2x)} = \quad \text{Product Property of Square Roots} \]
\[ \sqrt{90x^5} = \quad \text{Multiply} \]
\[ \sqrt{9x^4 \cdot 10x} = \quad \text{Factor out each perfect square} \]
\[ \sqrt{9x^4 \cdot \sqrt{10x}} = \quad \text{Use the Product Property of Square Roots} \]
\[ 3x^2\sqrt{10x} \]

Sometimes you have to use the distributive property to multiply radical expressions.

Example:
\[ \sqrt{14}(\sqrt{7} - 5\sqrt{2}) \]
\[ \sqrt{14}\sqrt{7} - \sqrt{14}(5\sqrt{2}) = \quad \text{Apply the distributive property} \]
\[ \sqrt{14}(7) - 5\sqrt{14}(2) = \quad \text{Use the Product Property of Square Roots} \]
\[ \sqrt{7} \cdot 2 \cdot 7 - 5\sqrt{7} \cdot 2 \cdot 2 \]
\[ \sqrt{49(2)} - 5\sqrt{7(4)} = \quad \text{Factor out each perfect square from the radicand if possible} \]
\[ \sqrt{49\sqrt{2}} - 5\sqrt{7\sqrt{4}} = \quad \text{Use the product property of square roots} \]
\[ 7\sqrt{2} - 5 \cdot 2\sqrt{7} = \quad \text{Simplify} \]
\[ 7\sqrt{2} - 10\sqrt{7} = \quad \text{Combine like radicands} \]

To multiply sums and differences of two radicals, use the same method that you use to multiply two binomials, FOIL.
You can use the Quotient Property of Square Roots to simplify a radical expression.

**The Quotient Property of Square Roots states:** For any real numbers $a$ and $b$, where $a \geq 0$ and $b \geq 0$, 
\[ \sqrt{a/b} = \sqrt{a}/\sqrt{b}. \]

A **radical expression is in simplest radical form** when the radicand has no perfect square factors other than one, the radicand has no fractions, and there are no radicals in the denominator of a fraction.

**Rationalizing the denominator** of a radical expression is a method used to eliminate radicals from the denominator. One way to rationalize the denominator is to multiply both the numerator and the denominator by the radical expression in the denominator.

**Examples:**

**Simplify** $\sqrt{75}/9 = \sqrt{75}/\sqrt{9} = \sqrt{25\cdot5}/\sqrt{9} = \sqrt{25}\cdot\sqrt{5}/\sqrt{9} = 5\sqrt{5}/3$

**Simplify** $\sqrt{28mn}/8n = \sqrt{28mn}/\sqrt{8} = \sqrt{28\cdot\sqrt{m}\cdot\sqrt{n}}/\sqrt{8} = \sqrt{7\cdot4\cdot\sqrt{mn}}/\sqrt{2\cdot4} = 2\sqrt{7mn}/2\sqrt{2} = \sqrt{7mn}/\sqrt{2}$, now we have to rationalize the denominator:

$\sqrt{7mn}/\sqrt{2} \cdot \sqrt{2}/\sqrt{2} = \sqrt{14mn}/2$

**Simplify** $3\sqrt{2x}/\sqrt{11y} =

3\sqrt{2x}/\sqrt{11y} \cdot \sqrt{11y}/\sqrt{11y} =

3\sqrt{2x} \cdot \sqrt{11y}/11y =

3\sqrt{22xy}/11|y|
Chapter 9-4 Solve Radical Equations

An equation that contains a variable within a radical is called a radical equation. Solving equations involves inverse operations. Since squaring and finding square roots are inverse operations to solve and equation with a radical expression you will need to square both sides of the equation.

In general, you "solve" equations by "isolating" the variable; you isolate the variable by "undoing" whatever had been done to it.

When you have a variable inside a square root, you undo the root by doing the opposite: squaring. For instance, given \( \sqrt{x} = 4 \), you would square both sides: \( (\sqrt{x})^2 = 4^2 \) so \( x = 16 \).

There are a couple of issues that frequently arise when solving radical equations. The first is that you must square sides, not terms. Here is a classic example of why this is so:

I start with a true equation and then square both sides:

\[
\begin{align*}
3 + 4 &= 7 \\
(3 + 4)^2 &= 7^2 \\
49 &= 49
\end{align*}
\]

...but if I square the terms on the left-hand side:

\[
\begin{align*}
3 + 4 &= 7 \\
3^2 + 4^2 &= 7^2 \\
9 + 16 &= 49 \\
25 &= 49
\end{align*}
\]

.................Oops!

The most common mistake that students make when solving radical equations is squaring terms instead of sides. Don't make this mistake! You should always remember to:

** SQUARE SIDES, NOT TERMS **

The other issue is that you will need to check your answers. You can always check your answers in a solved equation by plugging your answer back into the original equation and making sure that it fits.

The difficulty with radical equations is that you may have done every step correctly, but your answer may still be wrong. This is because the very act of squaring the sides can create solutions that never existed before.

For instance, I could say "\(-2 = 2\)", and you would know that this is false. But look what happens when I square both sides:

\[
\begin{align*}
(\cdot2)^2 &= 2^2 \\
4 &= 4
\end{align*}
\]

I started with something that was not true, squared both sides of it, and ended with something that was true. This is not good!

A more pertinent example would be this: \( \sqrt{x} = -3 \)

This "equation" is no more true than the "\(-2 = 2\)" "equation" above, because no positive square root can ever equal a negative number.

But suppose I hadn't noticed that this equation has no solution, and had proceeded to square both sides:
By squaring, \((\sqrt{x})^2 = (-3)^2\)

I created a solution ("\(x = 9\)") that had not existed before and is in fact not valid. But I won't discover this error unless I remembered to check my solution: \(\sqrt{9} = -3\)

\[ 3 = -3 \text{ is false} \]

So the actual answer for the equation \(\sqrt{x} = -3\) is "no solution".

We would say that \(x = 3\) is an extraneous solution because it does not satisfy the equation.
Chapter 9-5 The Pythagorean Theorem

The Pythagorean Theorem states: If a triangle is a right triangle, then the sum of the squares of the legs, a and b, equals the square of the hypotenuse, c.

\[ a^2 + b^2 = c^2 \]

It is often written in the form of the equation:

You can use the Pythagorean theorem to find the length of a missing side of a right triangle. You can use the converse of the Pythagorean Theorem to test whether a triangle is a right triangle.

Example:

Find the length of the missing side of the given right triangle.

\[ a^2 + b^2 = c^2 \]
\[ x^2 + (14)^2 = (50)^2 \]
\[ x^2 + 196 = 2500 \]
\[ x^2 = 2500 - 196 \]
\[ x^2 = 2304 \]
\[ \sqrt{x^2} = \sqrt{2304} \]
\[ x = 48 \]

A Pythagorean triple is a set of three positive whole numbers, a, b, and c, that can be the lengths of the sides of a right triangle. Some Pythagorean triples are \{3,4,5\}, \{8,15,17\} and \{7,24,25\}. To be a Pythagorean triple the whole numbers are the lengths of the sides of right triangles, so the numbers in the sets must satisfy the Pythagorean Theorem. The greatest number in the set will be the hypotenuse.

Example: Determine whether 11, 60, 61 is a Pythagorean triple.

\[ (11)^2 + (60)^2 = (61)^2 \]
\[ 121 + 3600 = 3721 \]
\[ 3721 = 3721 \text{ Yes, } \{11, 60, 61\} \text{ is a Pythagorean Triple} \]
Chapter 9-6 Distance in the Coordinate Plane

In this segment we will investigate three methods for finding the length of a line segment in a coordinate plane, depending on whether the line segment is horizontal, vertical or oblique. We will apply the distance formula which is derived from the Pythagorean Theorem to find the length of an oblique segment.

The length of a vertical line segment is the absolute value of the difference between the y-coordinates of its endpoints. You know you have a vertical line when the x-coordinates are the same for each point.
The distance between \((x_1, y_1)\) and \((x_2, y_2)\) is \(|y_2 - y_1|\)

The length of a horizontal line segment is the absolute value of the difference between the x-coordinates of its endpoints. You know you have a horizontal line when the y-coordinates are the same for each point.
The distance between \((x_1, y_1)\) and \((x_2, y_2)\) is \(|x_2 - x_1|\)

To find the length of a line segment that is neither vertical or horizontal, you can use the Pythagorean Theorem.

**The Distance Formula which is derived from the Pythagorean Theorem can be used to find the distance, d, between any two points \((x_1, y_1)\) and \((x_2, y_2)\).**

- Create a right triangle by drawing lines parallel to the axes through the points.
- The lines intersect at \((x_2, y_1)\)
- Let \(d\) represent the length of the hypotenuse
- \(|x_2 - x_1|\) subtract the x-coordinates to find the length of the horizontal leg
- \(|y_2 - y_1|\) subtract the y-coordinates to find the length of the vertical leg
- \(d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2\) Use the Pythagorean Theorem
- \(\sqrt{d^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\) Find the square root of both sides
- \(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\) The Distance Formula

**Example:** Find the distance in Simplest radical form between each set of points. (-3,8) and (-10,1)

\[
d = \sqrt{((-10)-(-3))^2 + (1-8)^2}\ 	ext{Use the distance formula}
\]
\[
d = \sqrt{(-7)^2 + (-7)^2}\ 	ext{Simplify}
\]
\[
d = \sqrt{49 + 49}\ 	ext{Simplify}
\]
\[
d = \sqrt{98}\ 	ext{Simplify}
\]
\[
d = \sqrt{49} \cdot 2\ 	ext{Simplify}
\]
\[
d = \sqrt{49} \cdot \sqrt{2}\ 	ext{Use the Product Property of Square Roots}
\]
\[
d = 7\sqrt{2}\ 	ext{Write the solution in simplest radical form}
\]