Chapter 8.1 – Common Monomial Factors

The greatest common monomial factor of two or more monomials is the product of all integer and variable factors that are common to those monomials.

To find the greatest common monomial factor:
1. Write the prime factorization of each coefficient. Choose the least power of each prime factor that appears in all coefficients. The GCF is the product of those prime factors.
2. Write the least power of each variable factor that appears in all monomials. The GCF is the product of those variable factors.
3. The greatest common monomial factor of the monomials is the product of these two results.

Example: Find the greatest common monomial factor of 36x^2 and 54x^3

Prime factorization of 36 = 4*9 = 2*2*3*3 = 2^2*3^2
Prime factorization of 54 = 6*9 = 2*3*3*3 = 2*3^3
GCF = 2*3^2 = 18
Variable factors of x^2 = x*x
Variable factors of x^3 = x*x*x
GCF = x*x = x^2

greatest common monomial factor of 36x^2 and 54x^3 is 18x^2

Factoring is the reverse process of multiplying. If a polynomial can be written as the product of factors, then it is factorable.

You can factor a polynomial by:
1. Finding the greatest common monomial factor of the polynomial.
2. Divide the polynomial by the greatest common monomial factor to find the other factor.
3. Write the polynomial as the product of the two factors.

Example: Factor 9x^2y^3 + 54x^3y
Prime factorization of 9 = 3*3 = 3^2
Prime factorization of 54 = 6*9 = 2*3*3*3 = 2 * 3 * 3^2

Variable factors: x^2 = x*x, y^3 = y*y*y
Variable factors: x^3 = x*x*x, y = y
Greatest common monomial factor: 3^2x^2y = 9x^2y

\[
\begin{align*}
9x^2y^3 + 54x^3y &= 9x^2y^3 + 54x^3y \\
&= 9x^2y^3 + 54x^3y \\
&= 9x^2y^3 + 54x^3y \\
&= 9x^2y(y^2 + 6x)
\end{align*}
\]

A polynomial is factored completely when when it is expressed as a product of one or more polynomials that cannot be factored further.

A polynomial whose terms do not have a common factor other than one is said to be a prime polynomial.
Chapter 8.2 – Factor Trinomials: $ax^2 + bx + c$, $a = 1$

When attempting to factor quadratics that have a leading coefficient of 1, we must focus on
the values of $b$ and $c$. The systematic approach for factoring involves factoring $c$ in as many
different ways as possible into pairs. Then, find the pair that has a sum that amounts to the
value of $b$.

In short, we are looking for a pair of numbers that satisfy two requirements at the same time:
they must have a product equal to $c$ and a sum equal to $b$. Use the examples below to help
clarify this technique.

When $c$ is positive:
If the middle term $bx$ is positive, both factors are positive.
If the middle term $bx$ is negative, both factors are negative.

When $c$ is negative:
The larger of these factors has the same sign as the middle term.

1. Let's start with $x^2 + 10x + 21$. Since $c = 21$, we will factor 21 in as many pairs as
possible. The pairs are: {1, 21} and {3, 7}. Now we must find a pair that has a sum
equal to $b$, which is 10. The first pair has a sum of 22, but this does not match our
value for $b$. The second pair has a sum of 10, and that is a match!

This means that this trinomial can be factored to $(x + 3)(x + 7)$.

2. Let's take another trinomial that has more possibilities, namely $x^2 - 2x - 24$. This
problem is more involved because $c$ has a lot of factors and because there are
negatives to juggle. If we make a list of all possible factors of -24, we get: {1, -24},
{2, -12}, {3, -8}, {4, -6}, {6, -4}, {8, -3}, {12, -2}, and finally {24, -1}. Our pairs
have sums of -23, -10, -5, -2, 2, 5, 10, and 23, respectively. Our fourth sum is a
match, so {4, -6} is the winning combination.

This means $x^2 - 2x - 24$ factors to $(x + 4)(x - 6)$.

If you have a trinomial with the leading term of $x^2$ in the form $x^2+bxy+cy^2$, consider $cy^2$ as
the constant term and $by$ as the coefficient of the linear term.
Chapter 8.3 – Factor trinomials: \( ax^2 + bx + c \), \( a \neq 1 \)

Here, you will practice factoring trinomials of the form \( ax^2 + bx + c \), where \( a \), \( b \), and \( c \) are integers.

The "factoring by grouping" method can be used to factor \( ax^2 + bx + c \).
This method is usually much more efficient than trial and error:

- Find numbers that multiply to \( ac \) and add to \( b \).
- Group and factor, as indicated in the example below.

**EXAMPLE:**
Factor \( 6x^2 - 13x - 5 \) by grouping.
Here, \( a = 6 \), \( b = -13 \), and \( c = -5 \).
Thus, we need to find numbers that multiply to \( ac = 6(-5) = -30 \) and that add to \( b = -13 \).
The numbers -15 and 2 work.

Then,
\[
6x^2 - 13x - 5 \quad \text{(original expression)}
\]
\[
= 6x^2 - 15x + 2x - 5 \quad \text{(the two numbers add to -13, so rewrite the middle term using these two numbers)}
\]
\[
= (6x^2 - 15x) + (2x - 5) \quad \text{(group together the first two terms and the last two terms)}
\]
\[
= 3x(2x - 5) + (1)(2x - 5) \quad \text{(factor 3x out of the first two terms)}
\]
\[
= (2x - 5)(3x + 1) \quad \text{(factor out (2x - 5))}
\]

Or, equivalently,
\[
6x^2 - 13x - 5 \quad \text{(original expression)}
\]
\[
= 6x^2 + 2x - 15x - 5 \quad \text{(the two numbers add to -13, so rewrite the middle term using these two numbers)}
\]
\[
= (6x^2 + 2x) + (-15x - 5) \quad \text{(group together the first two terms and the last two terms)}
\]
\[
= 2x(3x + 1) + (-5)(3x + 1) \quad \text{(factor 2x out of the first two terms, and -5 out of the last two terms)}
\]
\[
= (3x + 1)(2x - 5) \quad \text{(factor out (3x + 1))}
\]
Chapter 8.4 – Special Products and Factoring:

\[(a+b)^2 = a^2 + 2ab + b^2\] or \[(a-b)^2 = a^2 - 2ab + b^2\]

The square of a binomial is a perfect square trinomial. It is a special product.

\[
\begin{align*}
(a+b)^2 &= (a+b)(a+b) = a^2 + ab + ab + b^2 \\
&= a^2 + 2ab + b^2
\end{align*}
\]

\[
\begin{align*}
(a-b)^2 &= (a-b)(a-b) = a^2 - ab - ab + b^2 \\
&= a^2 - 2ab + b^2
\end{align*}
\]

The square of a consists of the square of the first term, plus or minus twice the product of both terms, plus the square of the last term:

\[
(a+b)^3 = a^2 + 2ab + b^2
\]

\[
(a-b)^3 = a^2 - 2ab + b^2
\]

Example: \((d+8)^2\)

\[
\begin{align*}
a &= d \\
b &= 8 \\
a^2 + 2ab + b^2 &= d^2 + 2(d)(8) + (8)^2 = d^2 + 16d + 64
\end{align*}
\]

\[
(y-11)^2
\]

\[
\begin{align*}
a &= y \\
b &= -11 \\
y^2 + 2(y)(-11) + (-11)^2 &= y^2 - 22y + 121
\end{align*}
\]

The rules for squaring binomials can be used to identify and factor perfect square trinomials.

A **trinomial is a perfect square if:**

- The first and last terms are perfect squares.
- The middle term is twice the product of one factor from the first term and one factor from the last term.

Example: Determine whether the trinomial is a perfect-square trinomial: \(25y^2 + 30y + 9\)

The first term \(25y^2\) and the last term \(9\) are perfect squares.
The middle term: \(2(5y\times3) = 2(15y) = 30y\)

Yes \(25y^2 + 30y + 9\) is a perfect square trinomial and can be factored as \((5y+3)(5y+3) = (5y+3)^2\)
Chapter 8.5 – Special Product and Factoring \((a+b)(a-b) = a^2-b^2\)

Another special product is the result of multiplying the sum of two terms by the difference of the same two terms. The outer and inner terms of the FOIL multiplication will be additive inverses so their sum will be zero. The product of the sum and the difference of the same two terms is the square of the first term minus the square of the second term. Such a product is called the **difference of two squares**.

Example: \((n+11)(n-11)\)

\[
a = n \\
b = 11 \\
a^2-b^2 = n^2 - 11^2 = n^2 - 121
\]

**The difference of two squares:**

\((a + b)(a - b) = a^2 - b^2\)

You can factor the difference of two squares as the product of two binomials that are the sum and the difference of the same two numbers. To determine whether a binomial is a difference of two squares, the first and second terms must be perfect squares.

Example: Is \(36m^2 - n^2\) the difference of two squares?

The first terms is a perfect square \((6m)(6m)\)
The second term is a perfect squares \((n)(n)\)

Factored \(36m^2 - n^2\) is \((6m+n)(6m-n)\) it is the difference of two squares.
Chapter 8.6 – Factor by Grouping

You can apply the distributive property to factor a polynomial with a common binomial factor in the
same way you can factor a polynomial with a common monomial factor.

Example: Factor $2x(x-5) - 3(x-5)$ common binomial factor is $(x-5)$

$(x-5)(2x-3)$ factor the common binomial factor out and apply the distributive property.

So in factored form $2x(x-5)-3(x-5) = (2x-3)(x-5)$

When a polynomial has four or more terms you can sometimes factor by grouping if there are two
groups of terms that have the same binomial factor.

Example: Factor $3x^3 - 6x^2 - 2x - 4$

Arrange in groups with common factors $(3x^3-6x^2) + (2x-4)$
Factor the GCF from each group: $3x^2(x-2) + 2(x-2)$
Apply the distributive property: $(3x^2+2)(x-2)$
So in factored form $3x^3 - 6x^2 - 2x - 4$ is $(3x^2+2)(x-2)$

Sometimes a common binomial factor cannot be found by grouping the first two terms and the second
two terms. You need to use the commutative and associative properties of equality to rearrange and
group terms differently.

Example: Factor $2r^5 + 2 + r^2 + 4r^3$
Start by putting the polynomial in standard form: $2r^5 + 4r^3 + r^2 + 2$
Now try to group the first two terms and the second two terms: $(2r^5+4r^3) + (r^2+2)$
Factor out a common GCF: $2r^3(r^2+2) + (r^2 + 2)$
Use the distributive property: $(2r^3+1)(r^2+2)$

So $2r^5 + 2 + r^2 + 4r^3$ in factored form is $(2r^3+1)(r^2+2)$. 
Chapter 8.7 Factor Completely

A polynomial is factored completely when it is expressed as a product of one or more polynomials that cannot be factored further. Not all polynomials can be factored.

To factor a polynomial completely:
• Identify and factor out the greatest common monomial factor.
• If the degree of the polynomial is 2 or greater, try other methods of factoring that are based on the number of terms in the polynomial.

<table>
<thead>
<tr>
<th>Number of Terms</th>
<th>Type of Factoring</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two</td>
<td>Difference of Two Squares</td>
<td>$16x^2 - 25 = (4x+5)(4x-5)$</td>
</tr>
<tr>
<td>Three</td>
<td>Perfect Square Trinomial</td>
<td>$x^2 + 12x + 36 = (x+6)^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x^2 - 12x + 36 = (x-6)^2$</td>
</tr>
<tr>
<td>Three</td>
<td>Product of Two Binomials</td>
<td>$2x^2 + x - 10 = (2x+5)(x-2)$</td>
</tr>
<tr>
<td>Four</td>
<td>Factor by Grouping</td>
<td>$x^3 + 7x^2 + 3x + 21 = (x+7)(x^2 +3)$</td>
</tr>
</tbody>
</table>

You can check your answer by multiplying.