Chapter 7.1 – Introduction to Polynomials

A **monomial** is an expression that is a number, a variable or the product of a number and one or more variables with nonnegative exponents. Monomials that are real numbers are called constants. Examples of monomials: \( n, \ -2x^2, \ abc, \ 7, \ 2x^2y^3z^4 \)

Monomials with the same variables to the same powers are called **like terms**, if monomials are like terms only their coefficients can differ.

A **polynomial** is a monomial or the sum of monomials. Each of the monomials is a term of the polynomial. To write a polynomial in simplest terms you must combine like terms.

The **degree of a monomial** is the sum of the exponents of its variables. A term that has no variable part is called a constant term or a constant. The degree of a constant term is 0. Remember that a variable without an exponent has an exponent of 1.

The **degree of a polynomial** is the greatest degree of any of its terms after it has been simplified.

**A polynomial in simplest terms can be classified by its number of terms.**
- Monomial – a polynomial with only one term
- Binomial – a polynomial with two terms
- Trinomial – a polynomial with three terms

**A polynomial in one variable can also be classified by its degree.**
- Linear – a polynomial in one variable with a degree of one.
- Quadratic – a polynomial in one variable with a degree of two.
- Cubic – a polynomial in one variable with a degree of three.

When a polynomial in one variable is in its simplest form and its terms are written in descending order from left to right the polynomial is said to be in **standard form**. The coefficient of the first term of a polynomial in standard form is called the **leading coefficient**.
Chapter 7.2 – Add and Subtract Polynomials

To add monomials algebraically, you must combine like terms.

To add polynomials, group like terms and then find the sum. You can add polynomials horizontally or vertically.

To add the polynomials \((3x^2 + 4x - 2) + (2x^2 - x + 3) + (x^2 + 4)\)

**Method 1:** Use a horizontal format.
Group and combine like terms:
\[
6x^2 + 3x + 5
\]

**Method 2:** Use a vertical format:
Align like terms in columns.
Add the columns separately.
\[
\begin{align*}
3x^2 & + 4x - 2 \\
2x^2 & - x + 3 \\
x^2 & + 4 \\
6x^2 & + 3x + 5
\end{align*}
\]

To find the additive inverse of a polynomial, find the opposite, or additive inverse, of each term. Remember that the sum of a number and its additive inverse is 0.
The additive inverse of \(-3x^3 - 7x^2 + 4x + 2\) is \(3x^3 + 7x^2 - 4x - 2\).

To subtract monomials algebraically, add the opposite of the monomial being subtracted.

To subtract polynomials algebraically, add the opposite of the polynomial being subtracted. You can subtract polynomials horizontally or vertically.

Subtract \((9y^2z + 11yz^2 - 15z^2) - (12y^2z - 3yz^2 + 2z^2)\)

**Method 1:** Use the horizontal format:
Group and combine like terms.
\[
-3y^2z + 14yz^2 - 17z^2
\]

**Method 2:** Use the vertical format:
Align like terms in columns and subtract the columns separately.
Subtract by adding the opposite.
\[
\begin{align*}
9y^2z + 11yz^2 - 15z^2 \\
-(12y^2z - 3yz^2 + 2z^2)
\end{align*}
\]
\[
\begin{align*}
-3y^2z + 3yz^2 + 2z^2 \\
-3y^2z + 14yz^2 - 17z^2
\end{align*}
\]
Chapter 7.3 Multiply a Polynomial by a monomial

To multiply two monomials algebraically, multiply the coefficients. Then multiply the variables using the Law of Exponents for Multiplication.

The Law of Exponents for Multiplication states: \(a^m \cdot a^n = a^{n+m}\)

We will learn some new Laws for exponents:

The Law of Exponents for Powers states \((a^m)^n = a^{mn}\)

The Law of Exponents for a Power of a Product states \((ab)^m = a^m b^m\), where \(m\) is an integer.

To multiply a polynomial by a monomial algebraically use the steps:
Apply the distributive property to distribute the monomial
Multiply the monomials to simplify

Example: Multiply \(3a^2b^3(-7ab^2 - 3a^2b)\)

Apply the distributive property: \(3a^2b(-7ab^2) + 3a^2b^3(-3a^2b)\)
Multiply: \(3(-7)(a^2)(a)(b^2)b^2 + 3(-3)(a^2)(a^2)(b^3)b\)
\(-21a^3b^5 - 9a^4b^4\)
Chapter 7.5 – Multiply Binomials

There are numerous ways to set up the multiplication of two binomials. The first three methods shown here work for multiplying ALL polynomials, not just binomials. All methods, of course, give the same answer.

Multiply \((x + 3)(x + 2)\)

1. "Distributive" Method: The most universal method. Applies to all polynomial multiplications, not just to binomials.

Start with the first term in the first binomial - the circled blue \(X\). Multiply (distribute) this term times EACH of the terms in the second binomial.

\[
(x + 3)(x + 2) = x \cdot x + x \cdot 2
\]

Now, take the second term in the first binomial - the circled red +3 (notice we take the sign also). Multiply this term times EACH of the terms in the second binomial.

\[
(x + 3)(x + 2) = 3 \cdot x + 3 \cdot 2
\]

Add the results: \(x \cdot x + x \cdot 2 + 3 \cdot x + 3 \cdot 2\)

\[
x^2 + 2x + 3x + 6
\]

\[
x^2 + 5x + 6 \quad \text{Answer}
\]

Do you see the "distributive property" at work?
(x + 3)(x + 2) = x(x + 2) + 3(x + 2)

2. "Vertical" Method: This is a vertical "picture" of the distributive method. This style applies to all polynomial multiplications.

\[
\begin{array}{ccc}
x + 2 \\
x + 3 \\
\hline
x^2 + 2x \\
3x + 6 \\
\hline
x^2 + 5x + 6
\end{array}
\]

multiply “x” from bottom terms times “x+2”

multiply “3” from the bottom terms times “x+2”

add like terms

For Binomial Multiplication ONLY!

"FOIL" Method: multiply First Outer Inner Last

The words/letters used to describe the FOIL process pertain to the distributive method for multiplying two binomials. These words/letters do not apply to other multiplications such as a binomial times a trinomial.

\[
\begin{align*}
\text{F: } & (x + 3)(x + 2) \\
\text{O: } & (x + 3)(x + 2) \\
\text{I: } & (x + 3)(x + 2) \\
\text{L: } & (x + 3)(x + 2) \\
(x + 3)(x + 2) & = x^2 + 2x + 3x + 6 \\
& = x^2 + 5x + 6
\end{align*}
\]

The drawback to using the FOIL lettering is that it ONLY WORKS on binomial multiplication.
Chapter 7.6 – Multiply Polynomials

To multiply polynomials you can use the the **distributive property** that we learned when multiplying binomials.

\[
(n^2 + 2n + 6)(2n^2 - 3n - 3)
\]
\[
n^2(2n^2 - 3n - 3) + 2n(2n^2 - 3n - 3) + 6(2n^2 - 3n - 3)
\]
\[
n^2(2n^2) + n^2(-3n) + n^2(-3) + 2n(2n^2) + 2n(-3n) + 2n(-3) + 6(2n^2) + 6(-3n) + 6(-3)
\]
\[
2n^4 - 3n^3 - 3n^2 + 4n^3 - 6n^2 - 6n + 12n^2 - 18n - 18
\]
\[
2n^4 + n^3 + 3n^2 - 24n - 18
\]

To multiply polynomials you can **multiply vertically** like we learned when multiplying binomials.

\[
\begin{array}{c}
2n^2 - 3n - 3 \\
n^2 + 2n + 6 \\
2n^4 - 3n^3 - 3n^2 \\
4n^3 - 6n^2 - 6n \\
12n^2 - 18n - 18 \\
2n^4 + n^3 + 3n^2 - 24n - 18
\end{array}
\]

Or you can use a **tabular method** when multiplying polynomials.

1. Draw a table as shown below. Write the first polynomial on the left and the second polynomial above the table.
2. Multiply the monomials in rows and columns to complete the table.
3. Find the sum of the monomial products from the table by combining like terms.
4. Simplify.

\[
\begin{array}{ccc}
n^2 & 2n^2 & - 3n \\
2n & - 3n^3 & - 3 \\
6 & 4n^3 & - 6n^2 \\
& - 6n & - 6n
\end{array}
\]
\[
\begin{array}{ccc}
12n^2 & -18n & -18 \\
\end{array}
\]

\[
2n^4 + (– 3n^3 + 4n^3) + (– 3n^2 - 6n^2 + 12n^2) + (– 6n - 18n) - 18 \\
2n^4 + n^3 + 3n^2 - 24n - 18
\]
Chapter 7.7 – Divide a Polynomial by a Monomial

You can use the Law of Exponents for Division to simplify quotients containing monomials.

The Law of Exponents for Division states: For \( a \neq 0 \) and \( m \) and \( n \) as integers \( \frac{a^m}{a^n} = a^{m-n} \)

To divide monomials:
First divide the coefficients
Next divide by subtracting the exponents of the variable factors with the same base.
Then multiply the quotients obtained in the above steps.

To divide a polynomial by a monomial:
Divide each term of the polynomial by the monomial divisor.
\[
\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}
\]
The number of terms in the polynomial dividend equals the number of terms in the quotient.

Dividing by a monomial is the same as multiplying by its reciprocal.

You can use the definition of an exponent and the Law of Exponents for Powers to raise a quotient to a power. The Law of Exponents for a Power of a Quotient states: \( (a/b)^m = a^m/b^m \) where \( b \neq 0 \).

******* DO TRY THESE PROBLEMS AS A CLASS*******
First, I set up the division:

For the moment, I'll ignore the other terms and look just at the leading \(x\) of the divisor and the leading \(x^2\) of the dividend.

\[
x + 1 \longdiv{x^2 - 9x - 10}
\]

If I divide the leading \(x^2\) inside by the leading \(x\) in front, what would I get? I'd get an \(x\). So I'll put an \(x\) on top:

\[
x + 1 \longdiv{x^2 - 9x - 10} \quad \frac{x}{x + 1}
\]

Now I'll take that \(x\), and multiply it through the divisor, \(x + 1\). First, I multiply the \(x\) (on top) by the \(x\) (on the "side"), and carry the \(x^2\) underneath:

\[
x + 1 \longdiv{x^2 - 9x - 10} \quad \frac{x}{x + 1}
\]

Then I'll multiply the \(x\) (on top) by the \(1\) (on the "side"), and carry the \(1x\) underneath:

\[
x + 1 \longdiv{x^2 - 9x - 10} \quad \frac{x}{x^2 + 1x}
\]

Then I'll draw the "equals" bar, so I can do the subtraction.

To **subtract** the polynomials, I **change all the signs** in the second line...

\[
x + 1 \longdiv{x^2 - 9x - 10} \quad \frac{x}{-x^2 + 1x}
\]

...and then I add down. The first term (the \(x^2\)) will cancel out:

\[
x + 1 \longdiv{x^2 - 9x - 10} \quad \frac{x}{-x^2 + 1x}
\]

\[
\begin{align*}
-10x
\end{align*}
\]

I need to remember to carry down that last term, the "subtract ten", from the dividend:
Now I look at the $x$ from the divisor and the new leading term, the $-10x$, in the bottom line of the division. If I divide the $-10x$ by the $x$, I would end up with a $-10$, so I'll put that on top:

$$\begin{array}{c}
x + 1) x^2 - 9x - 10 \\
\underline{-x^2 + 10x} \\
-10x - 10 \\
\end{array}$$

Now I'll multiply the $-10$ (on top) by the leading $x$ (on the "side"), and carry the $-10x$ to the bottom:

$$\begin{array}{c}
x + 1) x^2 - 9x - 10 \\
\underline{-x^2 + 10x} \\
-10x - 10 \\
\end{array}$$

...and I'll multiply the $-10$ (on top) by the $1$ (on the "side"), and carry the $-10$ to the bottom:

$$\begin{array}{c}
x + 1) x^2 - 9x - 10 \\
\underline{-x^2 + 10x} \\
-10x - 10 \\
-10x - 10 \\
\end{array}$$

I draw the equals bar, and change the signs on all the terms in the bottom row:

$$\begin{array}{c}
x + 1) x^2 - 9x - 10 \\
\underline{-x^2 + 10x} \\
-10x - 10 \\
+10x + 10 \\
\end{array}$$

Then I add down:

$$\begin{array}{c}
x + 1) x^2 - 9x - 10 \\
\underline{-x^2 + 10x} \\
-10x - 10 \\
+10x + 10 \\
0 \\
\end{array}$$

Then the solution to this division is: $x - 10$

Since the remainder on this division was zero, the division came out "even". When you do regular division with numbers and the division comes out even, it means that the number you divided by is a factor of the number you're dividing. For instance, if you divide 50 by 10, the answer will be a nice neat "5" with a zero remainder, because 10 is a factor of 50. In the case of the above polynomial division, the zero remainder tells us that $x + 1$ is a factor of $x^2 - 9x - 10$, which you can confirm by factoring the original quadratic dividend, $x^2 - 9x - 10$. 


Chapter 7.8 – Divide Polynomials Using Long Division

You can use long division to divide a polynomial by a binomial:
To find the first term of the quotient, divide the first term of the dividend by the first term of the divisor.
To find the next term of the quotient, divide the first term of the partial dividend by the first term of the divisor.

When using long division it is very important that you align terms with the same degree before you subtract.

The dividend must contain every possible power of the variable. If one is missing, you need to insert place holders for any missing variables (terms) in the dividend by using 0 as the coefficient. For example: \( (y^3 + 15) / (y - 5) \)
You would have to rewrite the dividend as \( y^3 + 0y^2 + 0y + 15 \) before you could divide.

Before dividing you must make sure all polynomials are in standard form. This means that the exponents for the variable must be listed in decreasing order.

That is \( (16y + 12y^2 + 5) / (5 + 6y) \) should be rewritten as \( 12y^2 + 16y + 5 / 6y + 5 \)

When you divide a polynomial by a binomial and you do not have a remainder then you know that the binomial you divided by is a factor of the polynomial.